MATH 10550, EXAM 2 SOLUTIONS

1. Find an equation for the tangent line to

$$f(x) = \sin x - \cos x$$

at the point $(\frac{\pi}{4}, 0)$.
Solution.

$$f'(x) = \cos(x) + \sin(x)$$

Thus, $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$ which is the slope of the tangent line at $\left(\frac{\pi}{4}, 0\right)$. Thus $y = \sqrt{2}(x - \frac{\pi}{4}) = \sqrt{2}x - \frac{\sqrt{2}\pi}{4}$ is the tangent line.

2. Find the derivative of

$$y = \sin(\sqrt{1+x^2}).$$

Solution.

$$\begin{aligned} y' &= \cos(\sqrt{1+x^2})\,\frac{d}{dx}(\sqrt{1+x^2}) = \cos(\sqrt{1+x^2})\,\frac{1}{2}(1+x^2)^{-1/2}2x \\ &= \frac{x\cos(\sqrt{1+x^2})}{\sqrt{1+x^2}}. \end{aligned}$$

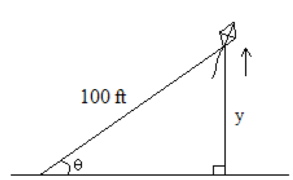
3. Find y' if

$$y^5 + x^2y^3 = 1 + x^4y.$$

Solution.

$$\begin{aligned} 5y^4y' + 3x^2y^2y' + 2xy^3 &= 4x^3y + x^4y' \\ y'(5y^4 + 3x^2y^2 - x^4) &= 4x^3y - 2xy^3 \\ y' &= \frac{2xy(2x^2 - y^2)}{5y^4 + 3x^2y^2 - x^4} \end{aligned}$$

4. A boy is flying a kite and has let 100 feet of string out. He does not let anymore string out and does not reel any more in. The wind is blowing the kite up higher, and the angle between the string of the kite and the horizontal is increasing at a rate of 2 radians/second. How quickly is the kite rising when the angle is $\pi/4$?
Solution.



where $y =$ the height in feet of the kite. We know $\frac{d\theta}{dt} = 2$ rad/s, and want to know $\frac{dy}{dt}$ when $\theta = \pi/4$. We note that $\sin\theta = \frac{y}{100}$, so

$$\cos(\theta)\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt}$$

Plugging in what we know gives us $\frac{dy}{dt} = 100\sqrt{2}$ ft/s when $\theta = \pi/4$.

5. Use linear approximation to estimate $\sqrt[5]{(1.005)^8}$.

Solution. If $f(x) = \sqrt[5]{x^8}$, then we are interested in $f(1.005)$. Now $f'(x) = \frac{8}{5}x^{3/5}$, so for x near 1, we have

$$f(x) \approx f(1) + f'(1)(x-1) = 1 + \frac{8}{5}(x-1).$$

Thus, $\sqrt[5]{(1.005)^8} \approx 1 + \frac{8}{5}(0.005) = 1.008$.

6. Find the absolute maximum and minimum of

$$f(x) = x - 4\sqrt{x}$$

on the closed interval $[0, 9]$, if they exist.

Solution. Note that f is continuous on the interval. The absolute maximum and minimum of a continuous function on a closed interval occur at critical points or the endpoints and are guarenteed to exist. Since

$$f'(x) = 1 - \frac{4}{2\sqrt{x}}$$

the function has critical points $x = 0$ and $x = 4$. We calculate $f(0) = 0, f(4) = -4$, and $f(9) = -3$, so f has its absolute maximum at $(0, 0)$ and its absolute minimum at $(4, -4)$.

7. Let

$$f(x) = x^3 + x - 1.$$

Determine whether or not f satisfies the hypothesis of the Mean Value Theorem on the interval $[0, 2]$. If it does, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution. Since f is a polynomial, it is continuous on $[0, 2]$ and differentiable on $(0, 2)$, and hence satisfies the hypothesis of the Mean Value Theorem. We calculate that $f'(x) = 3x^2 + 1$, $f(0) = -1$, and $f(2) = 9$. The Mean Value Theorem concludes that there exists at least one c in the interval $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{9 + 1}{2} = 5.$$

Since $f'(c) = 3c^2 + 1$, we are looking for numbers $0 < c < 2$ such that $3c^2 + 1 = 5 \implies 3c^2 = 4 \implies c^2 = \frac{4}{3} \implies c = \pm\frac{2}{\sqrt{3}}$. Since $-\frac{2}{\sqrt{3}}$ is not in $[0, 2]$, the only number that satisfies the conclusion of the Mean Value Theorem is $c = \frac{2}{\sqrt{3}}$.

8. Let

$$f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x.$$

Find the intervals where f is concave up.

Solution.

$$f'(x) = x^4 - 2x^2 + 1$$
$$f''(x) = 4x^3 - 4x = 4x(x^2 - 1).$$

f is concave up when $f'' > 0$. Note that $0 = f''(x)$ when $x = 0$, $x = -1$, and $x = 1$, and $f''(x)$ continuous. By the IVT we need only calculate $f''(x)$ for one value of x from each of the intervals $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$. Thus, since $f''(-2) = -24 < 0$, $f''\left(-\frac{1}{2}\right) = \frac{3}{2} > 0$, $f''\left(\frac{1}{2}\right) = -\frac{3}{2} < 0$, and $f''(2) = 24 > 0$ we conclude that f is concave up on $(-1, 0) \cup (1, \infty)$.

9. Evaluate the limit

$$\lim_{x\to\infty} \frac{2 - 3x^2 + x^3}{6x^3 + 5x^2 + 4x}$$

Solution. Divide the numerator and the denominator by x^3 and use that $\lim_{x\to\infty} \frac{1}{x^n} = 0$ if $n > 0$:

$$\lim_{x\to\infty} \frac{\frac{2}{x^3} - \frac{3}{x} + 1}{6 + \frac{5}{x} + \frac{4}{x^2}} = \frac{1}{6}.$$

10. Consider the function

$$f(x) = \frac{x^2}{x^2 + 9}.$$

One of the following statements is true. Which one?

Solution. To see if the function has a horizontal asymptote, we look at

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2 + 9} = \lim_{x\to\pm\infty} \frac{1}{1 + \frac{9}{x^2}} = 1.$$

Hence, f has a horizontal asymptote at the line $y = 1$. Now looking at max and min,

$$f'(x) = \frac{2x(x^2+9) - 2x^3}{(x^2+9)^2} = \frac{18x}{(x^2+9)^2}.$$

So f has a critical point at $x = 0$. Furthermore, $f'(-1) < 0$ and $f'(1) > 0$, and thus f has a global minimum at $x = 0$ by the First Derivative Test.

11. The position of a particle is given by

$$s(t) = t^2(9-t), \qquad \text{for } 0 \leq t \leq 9.$$

(a) When is the particle moving in the positive direction?

Solution. The particle is moving in a positive direction when its velocity is positive.

$$v(t) = s'(t) = 18t - 3t^2 = 3t(6-t)$$

is 0 when $t = 0$ or $t = 6$, so since $v(-1) < 0$, $v(1) > 0$, and $v(7) < 0$ the particle is moving in the positive direction only when t is in the interval $(0, 6)$.

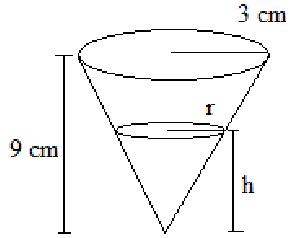
(b) What is the total distance travelled between $t = 0$ seconds and $t = 9$ seconds?

Solution. Since the particle turns around at $t = 6$,

$$\text{Total Distance} = |f(6) - f(0)| + |f(9) - f(6)| = 108 + 108 = 216$$

12. A paper cup has the shape of a cone with height 9 cm and radius 3 cm at the top of the cup. If water is poured into the cup at a rate of 2 cm^3/sec., how fast is the water level in the cup rising when the water is 3 cm deep? Hint: The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.

Solution.



where h is the height of the water in the cup and r is the radius of the cup at the top of the water. We know that $\frac{dV}{dt} = 2$ cm^3/s and want to know $\frac{dh}{dt}$ when $h = 3$ cm. From similar triangles we have that $\frac{r}{h} = \frac{3}{9}$, so $r = \frac{1}{3}h$. Hence $V = \frac{\pi}{27}h^3$ and so we have

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

Plugging in what we know gives us that $\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt} = \frac{2}{\pi}$ cm/s when $h = 3$ cm.

13. Let

$$f(x) = 3x^{2/3} - x.$$

(a) What are the critical numbers for f?

Solution.

$$f'(x) = 2x^{-1/3} - 1$$

is undefined when $x = 0$ and zero when $x = 8$, so $x = 0$ and $x = 8$ are the critcal points

(b) On what intervals is the function increasing?

Solution. $f'(-1) = -3 < 0$, $f'(1) = 1 > 0$, $f'(27) = -\frac{1}{3} < 0$, so the function is increasing only on the interval $(0, 8)$

(c) What are the local max?

Solution. Since f' switches from positive to negative at $x = 8$, f has a local maximum at $x = 8$ by the First Derivative Test. $f(8) = 4$

(d) What are the local min?

Solution. Since f' switches from negative to positive at $x = 0$, f has a local minimum at $x = 0$ by the First Derivative Test. $f(0) = 0$

(e) Is there a global max or min?

Solution. Since the highest power of x in $f(x)$ is 1, the term $-x$ dominates for large x, and thus the function will have the same behavior as $-x$; that is

$$\lim_{x\to\infty} f(x) = -\infty, \quad \text{and} \lim_{x\to-\infty} f(x) = \infty.$$

Or alternately, we can write $f(x) = x^{2/3}(3-x^{1/3})$. Then since $\lim_{x\to\pm\infty} x^{2/3} = \infty$, and $\lim_{x\to\pm\infty} 3 - x^{1/3} = \mp\infty$, we have that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^{2/3}(3 - x^{1/3}) = \mp\infty.$$

Thus f has no global max or global min.